

Calculating Distance in the Presence of Locally Varying Anisotropy

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A single stationary specification of anisotropy often cannot capture the complexities of a geologic site. In this situation, the anisotropy can be varied locally. Directions of continuity and/or variogram parameters can change depending on location within the domain being modeled. Kriging equations could easily be developed to use a local anisotropy specification within the Kriging neighborhood; however, this approach does not account for variation in anisotropy within the Kriging neighborhood. This paper presents an algorithm to determine the optimum path through a field of local anisotropy parameters that results in the highest covariance. Using the optimum path between two points increases covariance and generates estimates with lower variance. Although CPU intensive, the methodology can be implemented to reproduce complex curvilinear structures in Kriged or simulated maps. Synthetic examples will highlight the ability of this methodology to reproduce complex features that would otherwise be difficult to generate with traditional Kriging or sequential Gaussian simulation.

Introduction

A decision of stationarity is required for geostatistical techniques. If there are drastic differences in statistical parameters between areas of a geological site it could be subdivided into independent domains, each modeled separately. Occasionally a domain is further subdivided by rock-type or facies if such a subdivision is deemed important for the study objectives. In this case, geostatistical modeling is done on a by-facies basis. We will present a technique to explicitly deal with multiple directions of continuity within a single domain.

Inherent features within geological formations can be exploited to increase the accuracy of modeling including anisotropy, range of correlation, secondary variables, and others. If these characteristics are well understood, they can be transferred into modeling to improve performance. Consider a vein type deposit where the strikes of the veins are known and found to be quite variable over the deposit. In this situation it would be difficult to select a single variogram with one major direction of anisotropy. We propose to utilize a locally varying variogram in geostatistical modeling. At every location in the model a different variogram is selected by the modeler and Kriging is modified to use a different variogram.

In conventional geostatistical modeling, the relationship between any two locations in space is characterized by a variogram model. Angles and ranges define the anisotropy present in the variogram and it is applied in a stationary manner; that is, we use the same model for all locations within a spatial domain of interest. In this case, the straight-line covariance between any two locations in space is the maximum attainable covariance. Locally varying anisotropy (LVA) will be defined as a consistent variogram model in terms of nested structures and variance contributions, but with spatially varying range and direction parameters. In most cases the straight-line covariance will no longer be the maximum.

Some work has been done with locally varying variograms but it has not been fully developed. Often the variogram for the location being estimated (gray cell in Figure 1) is selected to be locally representative and the straight line path between the estimation location and surrounding data is used. Partitioning the path between the estimation location and surrounding data and calculating the anisotropic distance for every cell that the path intersects would give a more accurate measure of the covariance between points. Moreover, a non-linear path between points may result in a higher covariance when anisotropy is considered (dashed path in Figure 1).

This paper will discuss the optimization of the path between two points in the presence of a LVA field. The following paper, “Kriging and Simulation in the Presence of Locally Varying Anisotropy” will discuss the implementation of this algorithm in Kriging and simulation.

Background

An optimization algorithm will be developed for minimizing the anisotropic distance, which is equivalent to maximizing covariance, between two points in space when confronted with LVA. An LVA field is parameterized as a regular grid, with each cell characterized by three angles and two anisotropy ratios. Angles measure strike, α , dip, β , and plunge, γ , of the variogram ellipsoid and ratios, r_1 and r_2 , are the minor and vertical variogram ranges scaled by the maximum range. All five parameters can be combined into an anisotropic rotation matrix R in Equation 1 that is used in calculating the distance along a line within a particular block of an LVA field. Within the same LVA block the squared distance between two points, \mathbf{p}_0 and \mathbf{p}_1 , is calculated by Equation 2.

$$R = \begin{bmatrix} \cos \beta \cos \alpha & \cos \beta \sin \alpha & -\sin \beta \\ \frac{1}{r_1}(-\cos \gamma \sin \alpha + \sin \gamma \sin \beta \cos \alpha) & \frac{1}{r_1}(\cos \gamma \cos \alpha + \sin \gamma \sin \beta \sin \alpha) & \frac{1}{r_1}(\sin \gamma \cos \beta) \\ \frac{1}{r_2}(\sin \gamma \sin \alpha + \cos \gamma \cos \beta \cos \alpha) & \frac{1}{r_2}(-\sin \gamma \cos \alpha + \cos \gamma \sin \beta \sin \alpha) & \frac{1}{r_2}(\cos \gamma \cos \beta) \end{bmatrix} \quad (1)$$

$$d^2(\mathbf{p}_0, \mathbf{p}_1) = (\mathbf{p}_1 - \mathbf{p}_0)^T R^T R (\mathbf{p}_1 - \mathbf{p}_0) \quad (2)$$

Previous work

To the authors’ knowledge, all past work has used the LVA for a local search only (Xu, 1996; Sullivan, Satchwell and Ferrax, 2007; Deutsch and Lewis 1992). In this framework, when determining the Kriging weights the variogram of the estimation location is applied to its local neighborhood. The variogram is assumed constant for each Kriging neighborhood. Consider estimating at the gray location in Figure 2; the north-west direction would be applied to the local area and the covariance between points would be calculated with anisotropy in this direction. This is an exaggerated example where the LVA field at the estimation location is drastically different than the surrounding LVA, but it highlights the limitations of applying the LVA field in this way. If the LVA changes smoothly over the field and changes are observed beyond the range of the variogram, considering a constant local anisotropy in each Kriging neighborhood may be reasonable.

Stroet and Snepvangers (2005) have recently proposed a variant of Kriging where the local anisotropy is automatically calculated from the available data. Using locally varying anisotropy they are able to accurately reproduce curvilinear structures in both indicator and continuous variables using an iterative image analysis technique. Unfortunately, this technique is currently limited to 2D and requires sufficient data to directly infer the varying anisotropy. If the data does not show all the curvilinear structures, the method is not successful. Often times in petroleum and mining applications the data does not show the curvilinear features because of large sample spacing but it is known qualitatively, based on expert knowledge, that the features exist. In this case, our proposed methodology can be used to reproduce the desired features.

Methodology

The domain that will be estimated must be exhaustively filled with range and direction parameters that define the variogram at every location. Currently, the implementation of this technique requires that the LVA field be provided by the user in a regular grid. It would be difficult to generate the LVA field from data and often the geological understanding of the deposit is used to determine the directions of continuity,

i.e. the strike of veins, directions of channels etc. This section will discuss how the minimum distance between points is determined with the user supplied LVA field.

Calculating the distance between points

The use of R in Equation 2 provides the anisotropic distance. Now consider the same two points but separated by several LVA blocks having different anisotropy parameters; the equation for the squared distance between these two points can be developed (Fig. 3). The line that runs between two points can be split into segments, each within a specific LVA block. The line from \mathbf{p}_0 to \mathbf{p}_1 can be defined:

$$\mathbf{p} = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0) = \begin{bmatrix} x_0 + t(x_1 - x_0) \\ y_0 + t(y_1 - y_0) \\ z_0 + t(z_1 - z_0) \end{bmatrix} \quad (3)$$

where t is a parameter defining any point along the vector $\mathbf{p}_1 - \mathbf{p}_0$. Assuming the LVA field is an axis aligned regular grid, we know all of the x , y and z intersection points between \mathbf{p}_0 and \mathbf{p}_1 . These can be used in Equation 3 to calculate all values of t for which there is an intersection point. Each segment then belongs to a single LVA block and the distance $\mathbf{p}_1 - \mathbf{p}_0$ is the sum of the lengths of those segments. For m segments between two points, the squared distance is calculated by Equation 4.

$$\begin{aligned} D^2(\mathbf{p}_0, \mathbf{p}_1) &= \sum_{j=1}^m (t_j - t_{j-1})^2 (\mathbf{p}_1 - \mathbf{p}_0)^T R_j^T R_j (\mathbf{p}_1 - \mathbf{p}_0) \\ &= \sum_{j=1}^m \Delta t_j^2 \mathbf{v}^T A_j \mathbf{v} \end{aligned} \quad (4)$$

$$\text{where } \Delta t_j^2 = (t_j - t_{j-1})^2, \mathbf{v} = \mathbf{p}_1 - \mathbf{p}_0, \text{ and } A_j = R_j^T R_j$$

In Equation 4, j represents a segment within a specific LVA block associated with the rotation matrix R_j . To find the minimum squared distance between \mathbf{p}_0 and \mathbf{p}_1 , the path must be altered. Control-points are added to the path (Fig. 3). For n control-points, Equation 4 can be re-written as Equation 5. The goal is to find the positions for the control-points such that Equation 5 is minimal. Minimization of the distance between two points can be accomplished with the guarded Newton method (Boyd and Vandenberghe, 2004). This method requires the Jacobian and Hessian matrices, which are defined by Equations 6 and 7 below.

$$D^2(\mathbf{p}_0, \mathbf{p}_1) = \sum_{i=1}^n \sum_{j=1}^m \Delta t_{j(i)}^2 \mathbf{v}_{(i)}^T A_{j(i)} \mathbf{v}_{(i)} \quad (5)$$

$$\frac{\partial D}{\partial \mathbf{v}_{(i)}} = 2 \sum_{j=1}^{m_{(i)}} \Delta t_{j(i)}^2 A_{j(i)} \mathbf{v}_{(i)} - 2 \sum_{j=1}^{m_{(i+1)}} \Delta t_{j(i+1)}^2 A_{j(i+1)} \mathbf{v}_{(i+1)} \quad i = 1, \dots, n \quad (6)$$

$$\frac{\partial^2 D}{\partial \mathbf{v}_{(i)}^2} = 2 \sum_{j=1}^{m_{(i)}} \Delta t_{j(i)}^2 A_{j(i)} + 2 \sum_{j=1}^{m_{(i+1)}} \Delta t_{j(i+1)}^2 A_{j(i+1)}$$

$$\frac{\partial^2 D}{\partial \mathbf{v}_{(i)} \partial \mathbf{v}_{(i+1)}} = -2 \sum_{j=1}^{m_{(i+1)}} \Delta t_{j(i+1)}^2 A_{j(i+1)} \quad i = 1, \dots, n \quad (7)$$

$$\frac{\partial^2 D}{\partial \mathbf{v}_{(i)} \partial \mathbf{v}_{(i-1)}} = -2 \sum_{j=1}^{m_{(i)}} \Delta t_{j(i)}^2 A_{j(i)}$$

In the guarded Newton method, the Jacobian and Hessian are used to calculate a Newton step, which is a vector that describes the spatial translation to be applied to the control-points. Calculation of the Newton step ($\Delta\mathbf{v}$) for iteration k is done by solving the system in Equation 8. A line search is used to calculate the magnitude (α) of the step taken in Equation 9. Implementing a line search prevents translation of control-points into undesirable configurations, such as intersecting loops, or into positions beyond the LVA field. More control points are added and translated according to the Newton step and the process repeats until no improvement in distance can be attained.

$$H \cdot \Delta\mathbf{v} = -J \quad (8)$$

$$\mathbf{v}^{k+1} = \mathbf{v}^k + \alpha \cdot \Delta\mathbf{v}^k \quad (9)$$

The number of control points must account for the resolution of the regular grid over which the LVA field is defined. For a given resolution, two extreme cases may arise: (1) anisotropy is nearly constant everywhere in the field or (2) anisotropy is highly variable in orientation and range. In the first case, the minimum anisotropic distance could be found with very few control-points, possibly with only one. However, the second scenario may demand a control-point spacing equivalent to the LVA grid spacing.

To account for the range of LVA scenarios that exist between the two extreme cases, control-points are inserted exponentially as 2^x from one to a maximum, which is defined as the number of LVA blocks intersected between any two locations of interest. After each set of control-points are added, a new solution is found. Control-points are not added if the decrease in distance is less than ε (square-root of machine epsilon).

Minimizing the anisotropic distance between any two locations \mathbf{u} and \mathbf{v} in an LVA field is accomplished with the following algorithm:

1. Calculate initial distance between \mathbf{u} and \mathbf{v} , $D(\mathbf{u}, \mathbf{v})$.
2. Set iteration $k=0$, change in distance $\Delta D = D^k - D^{k-1}$, and the number of control-points $N_{cp}=0$.
3. While $\Delta D(\mathbf{u}, \mathbf{v}) > \varepsilon$ and $N_{cp} < max_{cp}$:
 - i. Insert 2^k control-points, where k is the current iteration, $N_{cp} = N_{cp} + 2^k$.
 - ii. Calculate Jacobian, Hessian and Newton step for all control-points.
 - iii. Calculate step parameter α using golden section search.
 - iv. Apply the Newton step.
 - v. Increment k .

Examples

Example 1 will highlight how the optimum non-linear path between two points changes as the LVA field changes. Example 2 will show the optimized paths between points in a geospatial setting where two data would be used to estimate at an unknown location.

Example 1

Consider the two points and LVA field in Figure 4. The LVA field is defined by the arrows shown on the plot and the different paths correspond to different anisotropy ratios. If the anisotropy ratio is 1:1 there is no benefit in altering the path and the straight-line path results in the maximum covariance. As the anisotropy ratio increases it becomes beneficial to consider the non-linear paths shown. Higher anisotropy ratios will result in paths that deviate further from the straight-line path. Table 1 shows the reduction in anisotropic distance accomplished with optimization. This will result in a higher and more realistic covariance between points.

Table 1: Differences in the optimized and straight-line paths between data in Figure 4.

<i>Anisotropy</i>	$h_{\text{Straight-line}}$	$h_{\text{Optimized}}$
1:1	8.7	8.7
1.33:1	10.86	10.82
2:1	15.34	15.31
4:1	29.25	29.21
5:1	36.32	31.42
10:1	71.94	59.66
20:1	143.52	121.69
1000:1	7170	6147

Example 2

Consider the data configuration and LVA fields in Figure 5. For Case 1 we will consider that the straight-line path is used between the points; however, the LVA field is used to calculate the covariance between points ($d_1+d_2+d_3$, Fig 5B). For Case 2 we will consider optimizing the path between points ($d_1+d_2+d_3$, Fig 5A). For both cases, the anisotropy ratio outside the channel will be 1:1 and inside the channel will be 10:1. Table 2 shows all the distances between points for both cases; there is a significant decrease in the anisotropic distance when the nonlinear path is used. In Kriging this will transfer to a larger covariance between points and will result in better estimates.

Table 2: Anisotropic distances between points for Example 2.

	<i>Case 1</i>	<i>Case 2</i>
	$h_{\text{straight-line}}$	$h_{\text{optimized}}$
$p_1 - p_2$	31.6	29.3
$p_1 - p_3$	60.7	50.8
$p_1 - p_4$	40.3	20.8
$p_2 - p_3$	7.8	7.4
$p_2 - p_4$	21.6	13.2
$p_3 - p_4$	12.8	6.2

Conclusions and Future Work

There are many geological sites that do not conform to the common assumption of stationarity. There are many solutions to this problem such as: trend modeling; domaining the data; facies modeling; using techniques that are not as dependant on the assumption of stationarity such as ordinary Kriging; or using an LVA model. Past work with LVA models has assumed that the local variogram at an estimation location applies to the neighborhood of that location. In order to depart from this assumption and consider that the LVA field varies between points, an algorithm to calculate the shortest non-linear anisotropic path between points is required. This is the algorithm that has been presented in this paper. The following paper “Kriging and Simulation in the Presence of Locally Varying Anisotropy” will implement this algorithm in Kriging and simulation.

There are two major areas of future work for this algorithm. First, other optimization techniques and approaches to finding the minimum anisotropic distance should be researched. The guarded Newton method implemented does not find the global minimum. This implies that after optimization, there may be a different path that results in a shorter distance between points. This is a common optimization problem and there are a number of solutions; the most common being to use a random restart. This would involve starting with a path other than the straight-line path between points. The drawback to this method is the added CPU time required to optimize the new starting path. The second area of future work is to improve the CPU speed of the algorithm. As will be discussed in “Kriging and Simulation in the Presence of Locally Varying Anisotropy” the CPU time of this algorithm becomes expensive.

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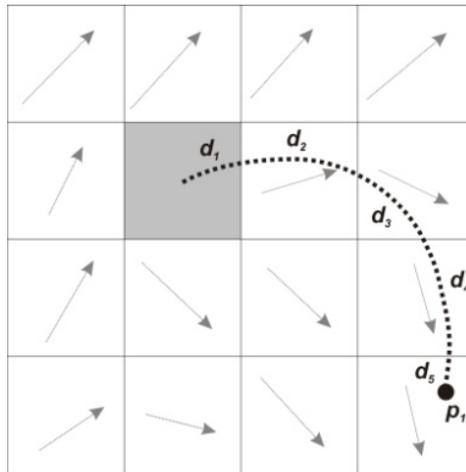


Figure 1: Calculating the true distance between the gray estimation location and a data point p_1 . The dashed path shows the optimized path.

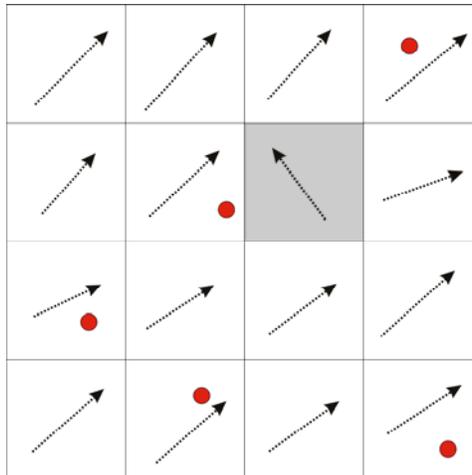


Figure 2: An example LVA field with the major direction of continuity shown as arrows and data as circles. Consider estimating the gray location without using the local change in anisotropy.

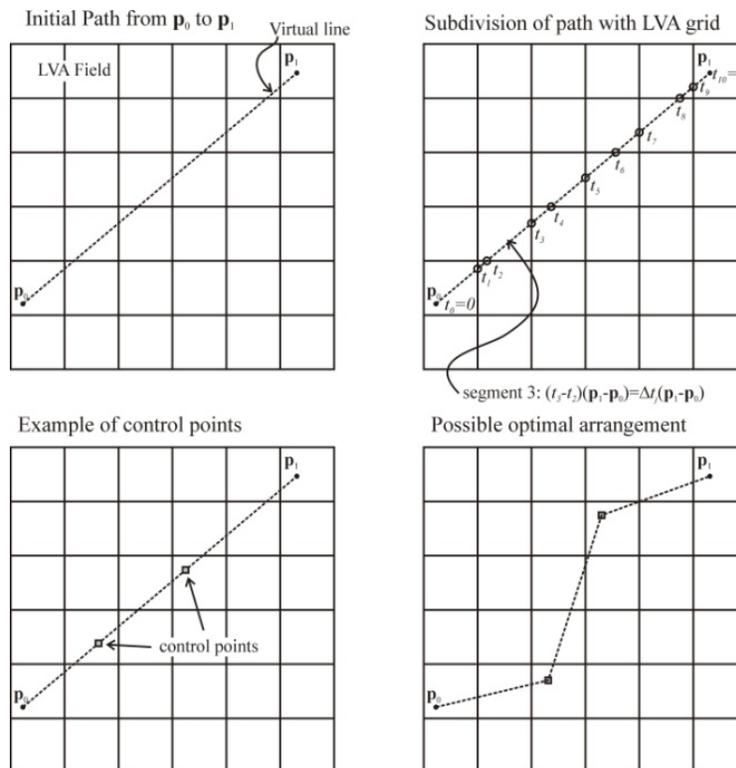


Figure 3: Components of minimizing the anisotropic distance.

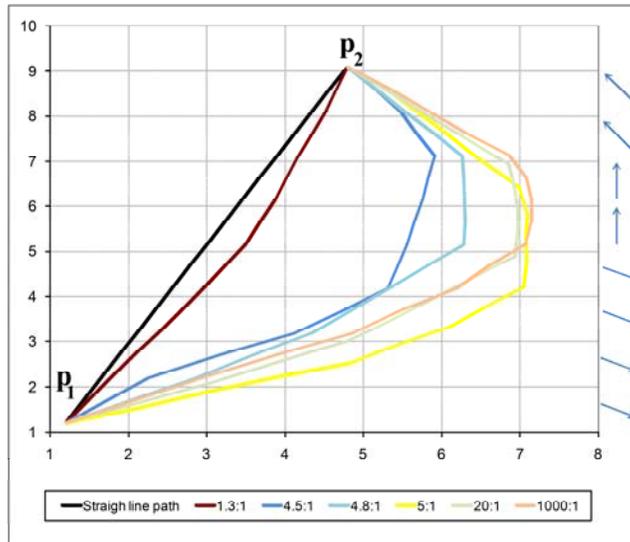


Figure 4: Optimized path between p_1 and p_2 . Different paths indicate different anisotropy ratios in the LVA field. The major direction of continuity for each row of cells is shown by the arrows to the right of the figure.

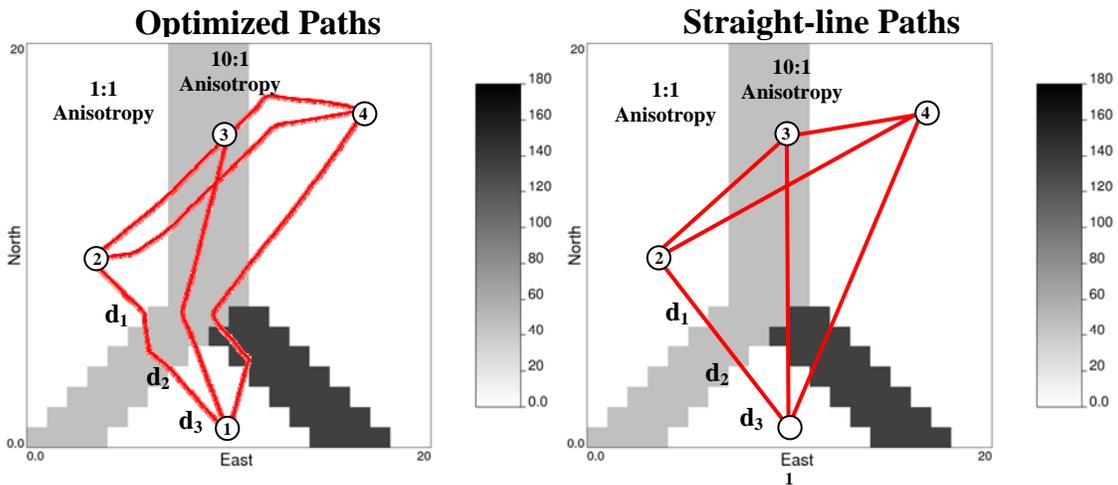


Figure 5: Location of data for Example 2. The background is colored based on the major direction of continuity of the LVA field and the anisotropy ratio in the channel is 10:1. *Sample calculation:* distance between points 1 and 2 is the sum of the segments $d_1+d_2+d_3$.